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2000 J. Phys. A: Math. Gen. 33 L387

(http://iopscience.iop.org/0305-4470/33/42/102)

J. Phys. A: Math. Gen. **33** (2000) L387–L391. Printed in the UK PII: S0305-4470(00)16418-2

LETTER TO THE EDITOR

Extremal driving as a mechanism for generating long-term memory

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Received 16 August 2000

Abstract. It is argued that systems whose elements are renewed according to an extremal criterion can generally be expected to exhibit long-term memory. This is verified for the minimal extremally driven model, which is first defined and then solved for all system sizes $N \geqslant 2$ and times $t \geqslant 0$, yielding exact expressions for the persistence $R(t) = [1+t/(N-1)]^{-1}$ and the two-time correlation function $C(t_w+t, t_w) = (1-1/N)(N+t_w)/(N+t_w+t-1)$. The existence of long-term memory is inferred from the scaling of $C(t_w+t, t_w) \sim f(t/t_w)$, denoting *aging*. Finally, we suggest ways of investigating the robustness of this mechanism when competing processes are present.

In recent years there has considerable progress in identifying the mechanisms responsible for long-term memory in glasses and other slowly relaxing systems, with processes such as domain coarsening and diffusion over random free energy landscapes now well established (see e.g. [1–3] and references therein). However, long-term memory has also been observed in a class of systems, namely the so-called *extremally driven* models, for which there is no obvious underlying mechanism. This is not a satisfactory state of affairs, as these models have applications covering a broad range of physically important situations, in particular to athermal and low temperature systems including granular media, flux creep, etc. One may reasonably ask how we can ever expect to understand the real systems if even their simplified models behave in a way that cannot be properly explained.

The defining characteristic of extremally driven models is that they are updated by identifying an 'active' region of the system, and *renewing* this region whilst leaving the remainder unaltered. The active subsystem is chosen according to some kind of *extremal* criterion; often it will be centred on the location of the minimum (or, equivalently, the maximum) of some spatially varying scalar variable, but other possibilities have been considered. Models that belong to this class include invasion percolation [4], the Bak–Sneppen model [5] and the granular shear model of Török *et al* [6], amongst others [7]. Recently, both the Bak–Sneppen and granular shear models have been found to exhibit *aging* [6, 8, 9], which indicates the existence of some form of long-term memory. The Bak–Sneppen model, along with many other extremally driven models, is *critical* in the sense that it has been placed by construction at a critical point (i.e. a continuous phase transition) of a broader phase diagram. By contrast, the granular shear model of Török *et al* is *not* critical, so if the same mechanism is responsible for both cases of aging, it cannot be due to any of the properties of the critical state.

In this Letter we demonstrate that extremal driving *by itself* is enough generate long-term memory, and claim that this is the true cause of the aging observed in the Bak–Sneppen and granular shear models. We further speculate that this mechanism is somewhat *robust* and that many other extremally driven models will also age; to the best of our knowledge, such behaviour has never been looked for in the other models in this class. The central part of this work is the solution of the simplest extremally driven model, which is shown to age in a similar manner to that observed in [6, 8]. Since the only mechanism in the model is the extremal driving, it is reasonable to infer that this is the cause of the long-term memory. A secondary aspect of this work is that the model can be *exactly* solved for all system sizes and times. This allows the finite size effects and transient behaviour to be explicitly calculated, which is rarely possible in systems exhibiting slow relaxation. It is also hoped that this work will help to extend the relationship between extremal statistics and glassy relaxation that was originally stressed by Bouchaud and Mézard [10].

The model to be studied here is defined as follows. The system consists of N elements which are each assigned a single scalar variable x_i, $i = 1, \ldots, N$, drawn from the fixed probability distribution function $p(x)$. For every time step $t \to t + 1$, the element with the smallest x_i in the system is selected and *renewed* by assigning it a new value of x_i, which is drawn from $p(x)$ as before. Non-degeneracy is assumed, i.e. no two x_i can take the same value, which is valid as long as N is finite and $p(x)$ contains no delta-function peaks. Selecting the maximum rather than the minimum would result in entirely equivalent behaviour and we henceforth restrict our attention to the minimum case only. This minimal model can be viewed as a one-dimensional version of the granular shear model [6], or equally as the Bak–Sneppen model without interactions [5].

Before solving the model, it is instructive to describe the emergence of long-term memory in qualitative terms. For this system, as for any system subjected to extremal driving, the typical values of x_i increase monotonically in time. This means that any recently renewed element is likely to have a smaller x_i than the bulk, and hence a shorter than average lifespan until it is again renewed. Correspondingly, elements that have not been renewed for some time will have a longer than average life expectancy. This separation between the shortest- and the longest-lived elements will become more pronounced as the system evolves and the average x_i in the bulk increases. Thus one might reasonably expect a broad distribution of relaxation times, and the possibility of long-term memory.

To put this picture into a more precise framework, let $P_t(S)$ be the probability to find the system in a state S after t updates, where $S = \{x_1, x_2 \ldots x_N\}$ (more formally the probability is $P_t(S) \prod_{i=1}^{N} \mathrm{d}x_i$ to simultaneously find the first variable in the range $(x_1, x_1 + \mathrm{d}x_1)$, the second in the range $(x_2, x_2 + \mathrm{d}x_2)$, etc.) Only one of the x_i changes value during a single time step. Thus to find $P_{t+1}(S)$ from $P_t(S)$, one must integrate over the region of phase space in which each of the x_i is the smallest and replace it with a value drawn from $p(x)$, i.e.

$$P_{t+1}(S) = \sum_{i=1}^{N} p(x_i) \int_{-\infty}^{\mathrm{m_i}} P_t(S^{(i)})\, \mathrm{d}x_i' \tag{1}$$

where the $\mathbb{R}^{N-1} \to \mathbb{R}$ function m_i is defined by

$$m_i = \min_{j \neq i} \{x_j\} \tag{2}$$

and $S^{(i)}$ is shorthand for $\{x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_N\}$. The factor of $p(x_i)$ on the right-hand side of (1) can be removed by making the change of variables $u_i = \int_{-\infty}^{x_i} p(z)\, \mathrm{d}z$, $0 \leqslant u_i \leqslant 1$, giving

$$Q_{t+1}(S) = \sum_{i=1}^{N} \int_0^{\mathrm{m_i}} Q_t(S^{(i)})\,\mathrm{d}u_i' \tag{3}$$

where $Q_t \prod_{i=1}^N \mathrm{d}u_i = P_t \prod_{i=1}^N \mathrm{d}x_i$. S, $S^{(i)}$ and m_i are here defined exactly as in (1) and (2), except with the x_i replaced by u_i. Scaling $p(x)$ out of the master equation in this manner reflects that, as in any extremally driven model, the dynamics depends only on the *order* of the x_i and *not* their relative spacings (note that there is no need to invoke universality to prove this result).

Before proceeding to solve the master equation (3), it is useful to state and prove the following identities. Firstly,

$$\int_0^{m_i} m_j^t\,\mathrm{d}u_i = \begin{cases} m_i^{t+1} & i = j \\ \dfrac{m_i^{t+1}}{t+1} & i \neq j. \end{cases} \tag{4}$$

The $i = j$ case is trivial (since m_j is independent of u_j), whereas for $i \neq j$ observe that $m_j \equiv u_i$ over the entire range of integration, from which the result follows. Another useful identity is

$$\int_{\mathcal{D}_i} m_j^t\,\mathrm{d}V = \begin{cases} \dfrac{(N-1)!(t+1)!}{(N+t)!} & i = j \\ \dfrac{(N-1)!\,t!}{(N+t)!} & i \neq j, \end{cases} \tag{5}$$

where $\mathcal{D}_i$ is the domain of space in which u_i is the smallest, and $\mathrm{d}V = \prod_{k=1}^N \mathrm{d}u_k$. This can be proven by considering in turn each of the $(N-1)!$ subregions of $\mathcal{D}_i$ defined by $u_i < u_{l_1} < u_{l_2} < \ldots < u_{l_{N-1}}$, where $l_k \neq i\ \forall k$. For each permutation of the l_k, the integral limits for each of the du can be inserted and the integration evaluated. The final result (5) then follows from summing over all the possible permutations.

The rescaled master equation (3) can be solved inductively from the initial state $Q_0 = 1$ by using the first identity (4), giving

$$Q_t(S) = \frac{(N+t-1)!}{t!\,N!} \sum_{i=1}^{N} \mathrm{m_i^t}. \tag{6}$$

That this is correctly normalized can be checked using the second identity (5). Since Q_t is symmetric in the m_i and therefore the u_i, the probability that any particular element in the system, say u_k, is the smallest at a given time t_w is just $1/N$. However, suppose it is known that u_k is *not* the smallest at t_w. Then $Q_{t_\mathrm{w}+1}$ can then be found by setting Q_{t_w} to zero in $\mathcal{D}_k$, renormalizing, and iterating (3) once. This three-step procedure can be repeated to find the following expression for $Q^k_{t_\mathrm{w}+t,t_\mathrm{w}}(S)$, defined as the probability to find the system in a state S at a time $t_\mathrm{w}+t$ given that u_k was not the minimum at any of the times $t_\mathrm{w}, t_\mathrm{w}+1, \ldots, t_\mathrm{w}+t-1$,

$$Q^k_{t_\mathrm{w}+t,t_\mathrm{w}}(S) = \frac{1}{N-1} \begin{pmatrix} N+t_\mathrm{w}+t-1 \\ N-1 \end{pmatrix} \sum_{i\neq k} m_i^{t_\mathrm{w}+t} \qquad t \geqslant 1. \tag{7}$$

The corresponding probability that u_k is the smallest, denoted here by $q^k_{t_\mathrm{w}+t,t_\mathrm{w}}$, can be calculated by integrating (7) over $\mathcal{D}_k$ and using (5),

$$q^k_{t_\mathrm{w}+t,t_\mathrm{w}} = \frac{1}{N+t_\mathrm{w}+t} \tag{8}$$

which is independent of k. This demonstrates that the probability of an element being renewed decreases with the time since it was *last* renewed, according to $q^k_{t_w+t,t_w} \sim t^{-1}$. Note that $q^k_{t_w+t,t_w}$ also decreases with t_w.

We are now in a position to calculate the physical quantities of interest, starting with the persistence $R(t)$ [11, 12]. $R(t)$ is defined as the probability that a randomly chosen element i has the same value of x_i at time t that it had at $t = 0$. Clearly, $R(0) = 1$ and $R(1) = (N-1)/N$. For $t \geqslant 2$, observe that $R(t)$ only decreases when an element is renewed for the first time, so $R(t+1) = (1 - q^k_{t,0})\, R(t)$ and hence from (8),

$$R(t) = R(1) \prod_{s=1}^{t-1} \left(1 - q^k_{s,0}\right) \tag{9}$$

$$= \frac{N-1}{N+t-1} \tag{10}$$

$$\sim \left(\frac{t}{N-1}\right)^{-\theta} \tag{11}$$

which defines the persistence exponent $\theta = 1$. The slow decay of $R(t)$ shows that a significant proportion of the system will remain in its initial state until arbitrarily late times, already suggesting some form of long-term memory. Note that there is no cut-off for finite system sizes; $R(t)$ asymptotically decays algebraically even for $N = 2$, as long as one averages over all possible initial conditions and histories.

The existence of aging can be most clearly expressed in terms of the two-time correlation function $C(t_w+t, t_w)$ between the state of the system at times t_w and t_w+t. A suitable $C(t_w+t, t_w)$ for this model is the probability that a randomly chosen element has the same value of x_i at t_w and $t_w + t$ (so $C(t, 0) \equiv R(t)$). As before, $C(t_w, t_w) = 1$, $C(t_w + 1, t_w) = (N-1)/N$ and

$$C(t_w + t, t_w) = C(t_w + 1, t_w) \prod_{s=t_w+1}^{t_w+t-1} \left(1 - q^k_{s,t_w}\right) \tag{12}$$

$$= \frac{N-1}{N} \left(\frac{N + t_w}{N + t_w + t - 1}\right) \qquad t \geqslant 1. \tag{13}$$

After a short transient this scales as

$$C(t_w + t, t_w) \approx \frac{N-1}{N} \left(1 + \frac{t}{t_w}\right)^{-1} \qquad \frac{t_w}{N} \gg 1. \tag{14}$$

That t and t_w only appear in the ratio t/t_w is what we mean by *aging*. Finally, note that in the limit $N \to \infty$, the N-dependence of (10) and (13) can be removed by renormalizing the timescale to $\tau \equiv t/N$, giving $R(\tau) = (1+\tau)^{-1}$ and $C(\tau_w + \tau, \tau_w) = (\tau_w + 1)/(\tau_w + \tau + 1)$, respectively.

We have now achieved what we set out to do, i.e. demonstrate that even the simplest extremally driven model has long-term memory, as evident from the aging of $C(t_w + t, t_w)$ in (14), and the slow decay of $R(t)$ (10). From this we infer that the extremally driven renewal is responsible for the aging observed in [6, 8, 9] and speculate that other extremal models, such as invasion percolation, may also age in a similar fashion. However, it should be stressed that not all extremally driven models will necessarily exhibit aging. Indeed, it is already known that including noise-like terms, by renewing randomly selected elements as well as the extremal one, introduces an exponential cut-off to the relaxation times and destroys the long-term memory (this situation is realized in the mean-field version of the Bak–Sneppen model, for instance [13]). Thus a useful step forward from this work might be to consider modified versions of the model, to see what physical processes may enhance or disrupt the effects of extremal driving. To this end, we tentatively suggest that the following modifications might be particularly worthy of investigation: introducing quenched disorder (by assigning each element its own generating distribution $p_i(x_i)$), allowing the values of x_i before and after renewal to be correlated, and letting the time step between updates be dependent on the x_i. It is especially hoped that these and other enhancements could be treated within the exact framework developed here.

In summary, we have argued that systems which are renewed according to an extremal criterion should be expected to exhibit long-term memory, and have supported this claim by showing that even the minimal extremally driven model ages. Expressions were found for the persistence $R(t)$ and the two-time correlation function $C(t_w + t, t_w)$ which corroborate these claims. Finally, we note that this work also constitutes an instance where the extremal properties of a system of correlated random variables can be exactly computed. To our knowledge, this makes it one of few such systems known [14–16].

This work was funded under UK EPSRC grant no. GR/M09674.

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